

Von Neumann Algebras: Double Commutant and Kaplansky Density

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Abstract

Von Neumann algebras are operator-algebraic analogues of measurable function algebras and form the natural setting for the spectral theory of unbounded operators, noncommutative integration, and quantum statistical mechanics. This paper presents a self-contained development of the operator topologies on $B(H)$, the equivalence of weak and strong closures on convex sets, von Neumann's double commutant theorem, and Kaplansky's density theorem. Proofs are included in full detail, emphasizing the role of locally convex topologies, the characterization of continuous linear functionals, and functional calculus techniques (Stone–Weierstrass and the Cayley transform).

Keywords: von Neumann algebra; weak operator topology; strong operator topology; commutant; double commutant theorem; Kaplansky density theorem; functional calculus; projections.

1 INTRODUCTION

Let H be a Hilbert space and $B(H)$ the C^* -algebra of bounded operators on H . A *von Neumann algebra* is a C^* -subalgebra of $B(H)$ that is closed in the strong (or, equivalently, weak) operator topology. These algebras arise as algebras of observables in quantum mechanics and as noncommutative measure spaces, where the weak/strong closures play the role of completion in measure. Two classical theorems underpin the subject: von Neumann's *double commutant theorem* characterizes von Neumann algebras algebraically as bicommutants, and Kaplansky's *density theorem* shows that the unit ball of a C^* -algebra is strongly dense in the unit ball of its strong closure. See [2, 5, 1] for standard background.

2 OPERATOR TOPOLOGIES AND BASIC FUNCTIONALS

We recall the locally convex framework used to define operator topologies.

2.1 Locally convex spaces and nets

A topological vector space is *locally convex* precisely when its topology is generated by a separating family of seminorms; equivalently, convergence is determined by those seminorms. Moreover, in locally

convex spaces, membership in the closure of a convex set can be tested against continuous linear functionals, using nets [1].

2.2 Strong and weak operator topologies

Definition 2.1. Let H be a Hilbert space. The *strong operator topology* (SOT) on $B(H)$ is the locally convex topology generated by seminorms $T \mapsto \|Tx\|$ for $x \in H$. The *weak operator topology* (WOT) is generated by seminorms $T \mapsto |\langle Tx, y \rangle|$ for $x, y \in H$.

Thus $T_\alpha \rightarrow T$ in SOT if $\|T_\alpha x - Tx\| \rightarrow 0$ for each $x \in H$, and $T_\alpha \rightarrow T$ in WOT if $\langle T_\alpha x, y \rangle \rightarrow \langle Tx, y \rangle$ for each $x, y \in H$.

2.3 Continuous linear functionals on $B(H)$

A key technical input is that WOT- and SOT-continuous linear functionals on $B(H)$ coincide and have an explicit form.

Theorem 2.2. A linear functional ϕ on $B(H)$ is WOT-continuous if and only if it is SOT-continuous if and only if there exist vectors $x_1, \dots, x_n, y_1, \dots, y_n \in H$ such that

$$\phi(T) = \sum_{k=1}^n \langle Tx_k, y_k \rangle, \quad T \in B(H).$$

Proof. We sketch the direction “SOT-continuous $\Rightarrow$ finite vector form”. Assume ϕ is SOT-continuous. Then there exist $x_1, \dots, x_n \in H$ and $C > 0$ such that $|\phi(T)| \leq C \max_{1 \leq k \leq n} \|Tx_k\|$ for all $T \in B(H)$. Consider the set

$$K_0 = \{(Tx_1, \dots, Tx_n) : T \in B(H)\} \subseteq H^n$$

and its norm closure $K \subseteq H^n$. Define $\psi : K_0 \rightarrow \mathbb{C}$ by $\psi(Tx_1, \dots, Tx_n) = \phi(T)$. The inequality above shows ψ is well-defined and bounded on K_0 , hence extends to a bounded linear functional on the Hilbert space K . By the Riesz representation theorem, there exists $(y_1, \dots, y_n) \in K$ such that $\psi(v_1, \dots, v_n) = \sum_{k=1}^n \langle v_k, y_k \rangle$ and substituting $v_k = Tx_k$ yields the required form. The other implications are standard

2.4 Weak and strong closures of convex sets

Theorem 2.3. If $Y \subseteq B(H)$ is convex, then its WOT-closure equals its SOT-closure.

Proof. By the locally convex closure criterion using nets, T lies in the WOT-closure of Y if there exists a net $T_\alpha \in Y$ such that $\phi(T_\alpha) \rightarrow \phi(T)$ for all WOT-continuous linear functionals ϕ . By Theorem 2.2, WOT- and SOT-continuous functionals coincide; hence the same net criterion characterizes SOT-closure. Since Y is convex, the characterization applies, giving equality of closures.

2.5 WOT-compactness of the unit ball

Theorem 2.4. The closed unit ball $(B(H))_1 = \{T \in B(H) : \|T\| \leq 1\}$ is compact in the WOT.

Proof. For each $x, y \in H$ define the compact disk $D_{x,y} = \{z \in \mathbb{C} : |z| \leq \|x\| \|y\|\}$. the product $\prod_{x,y} D_{x,y}$ is compact by Tychonoff. Map $T \in (B(H))_1$ to the point $(\langle Tx, y \rangle)_{x,y}$. This map is injective and a homeomorphism onto its image, and the image is closed, hence compact. Therefore $(B(H))_1$ is WOT-compact.

3 VON NEUMANN ALGEBRAS AND THE DOUBLE COMMUTANT THEOREM

3.1 Commutants

Definition 3.1. For $\mathcal{F} \subseteq B(H)$, its commutant is $\mathcal{F}' = \{T \in B(H) : TS = ST \forall S \in \mathcal{F}\}$. The double commutant is $\mathcal{F}'' = (\mathcal{F}')'$.

3.2 Bicommutants are von Neumann algebras

Proposition 3.2. If $\mathcal{F} \subseteq B(H)$ is self-adjoint (i.e. $\mathcal{F} = \mathcal{F}'$), then $\mathcal{F}'$ is a von Neumann algebra.

Proof. It is clear that $\mathcal{F}'$ is a unital $*$ -subalgebra. To show WOT-closure, let $T_\alpha \rightarrow T$ in WOT with each $T_\alpha \in \mathcal{F}'$. Fix $S \in \mathcal{F}$ and $x, y \in H$. Then

$$\langle (TS - ST)x, y \rangle = \langle TSx, y \rangle - \langle STx, y \rangle = \langle Sx, T^*y \rangle - \langle Tx, S^*y \rangle.$$

Using WOT-convergence and $T_\alpha S = ST_\alpha$, we obtain $\langle (TS - ST)x, y \rangle = \lim_\alpha \langle (T_\alpha S - ST_\alpha)x, y \rangle = 0$. Hence $TS = ST$ for all $S \in \mathcal{F}$, so $T \in \mathcal{F}'$. thus $\mathcal{F}'$ is WOT-closed, and therefore SOT-closed by Theorem 2.3.

3.3 Invariant and reducing subspaces via projections

Proposition 3.3. Let $M \subseteq H$ be a closed subspace and P the orthogonal projection onto M . An operator $T \in B(H)$ maps M into M if and only if $PTP = TP$.

Proof. If $(M) \subseteq M$, then for any $x \in H$, $Px \in M$ and hence $TPx \in M$, so $PTPx = TPx$ for all x , i.e. $PTP = TP$. Conversely, if $PTP = TP$ and $x \in M$, then $Px = x$ and $Tx = TPx = PTPx \in M$.

Corollary 3.4. With P as above, $TP = PT$ if and only if M is a reducing subspace for T (i.e. both $T(M) \subseteq M$ and $T^*(M) \subseteq M$).

Proof. If $TP = PT$, then $PTP = TP$, so $T(M) \subseteq M$ by Proposition 3.3. Taking adjoints gives $PT^* = T^*P$, hence $T^*(M) \subseteq M$. Conversely, if M reduces T then $PTP = TP$ and $PT^*P = T^*P$; taking adjoints yields $PT = TP$. $\square$

3.4 Approximation inside the bicommutant

Proposition 3.5. Let $A \subseteq B(H)$ be a unital self-adjoint subalgebra. For $x \in H$ and $T_0 \in A''$ there exists a sequence $(T_n) \subseteq A$ such that $\|T_n x - T_0 x\| \rightarrow 0$.

Proof. Let $M = \overline{\{Tx : T \in A\}} \subseteq H$. Since A is self-adjoint, M is reducing for each $T \in A$; hence the projection P onto M lies in A' (one shows $PT = TP$ using Corollary 3.4). Since $T_0 \in A''$, we have $T_0 P = PT_0$, so M also reduces T_0 . Because A is unital, $x \in M$ and thus $T_0 x \in M$. By definition of M , there are $T_n \in A$ with $T_n x \rightarrow T_0 x$. $\square$

A standard amplification argument extends the approximation simultaneously on finitely many vectors.

Corollary 3.6. Given $x_1, \dots, x_N \in H$ and $T_0 \in A''$, there exist $T_n \in A$ such that $\|(T_n - T_0)x_k\| \rightarrow 0$ for each $1 \leq k \leq N$.

Proof. Apply Proposition 3.5 to the diagonal amplification on $H^N : \Phi(T) = T \oplus \dots \oplus T$ and the vector $(x_1, \dots, x_N)$. $\square$

3.5 von Neumann's double commutant theorem

Theorem 3.7 (Double Commutant Theorem). *Let $A \subseteq B(H)$ be a C^* -subalgebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Proof. If $A = A''$, then A is SOT-closed because commutants are SOT-closed by Proposition 3.2, hence A is a von Neumann algebra.

Conversely, suppose A is SOT-closed. Since $A \subseteq A''$ always holds, it suffices to show A is SOT-dense in A'' . Let $T \in A''$ and let U be a basic SOT-neighborhood: $U = \{S \in B(H) : \|(S - T)x_k\| < \varepsilon, 1 \leq k \leq N\}$. By Corollary 3.6 there exists $S \in A$ with $S \in U$. Hence every SOT neighborhood of T meets A , so $T \in A^{-\text{SOT}} = A$. Thus $A'' \subseteq A$, proving $A = A''$.

4 KAPLANSKY DENSITY THEOREM

Let $A \subseteq B(H)$ be a C^* -subalgebra and let $B = A^{-\text{SOT}}$. Kaplansky's density theorem states that the closed unit ball of A is SOT-dense in the closed unit ball of B ; the proof relies on functional calculus and continuity properties in SOT.

4.1 Continuity facts for functional calculus

Proposition 4.1. *An operator $T \in B(H)$ is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in H$.*

Proof. If T is normal, then $T^*T = TT^*$ and $\|Tx\|^2 = \langle T^*Tx, x \rangle = \langle TT^*x, x \rangle = \|T^*x\|^2$. Conversely, if $\|Tx\| = \|T^*x\|$ for all x , then $\langle (T^*T - TT^*)x, x \rangle = \|Tx\|^2 - \|T^*x\|^2 = 0$, so $T^*T = TT^*$. $\square$

Proposition 4.2. *The map $T \mapsto T^*$ is SOT-continuous on bounded sets of normal operators.*

Proof. Let $T_\alpha \rightarrow T$ in SOT with T_α, T normal and $\sup_\alpha \|T_\alpha\| \leq r$. Fix $x \in H$. Using normality to replace norms of adjoints by norms of operators, compute

$$\|(T_\alpha^* - T^*)x\|^2 = \langle (T_\alpha - T)(T_\alpha^* - T^*)x, x \rangle = \|T_\alpha x\|^2 - \langle x, T_\alpha T^* x \rangle - \langle TT_\alpha^* x, x \rangle + \|Tx\|^2,$$

and observe each scalar term converges by SOT and boundedness (and hence WOT) on products. The right-hand side tends to 0, so $T_\alpha^* x \rightarrow T^* x$ for each x , i.e. $T_\alpha^* \rightarrow T^*$ in SOT.

Proposition 4.3. *Let f be continuous on $\mathbb{C}$. Then $T \mapsto f(T)$ is SOT-continuous on any bounded set of normal operators.*

Proof. Let $T_\alpha \rightarrow T$ in SOT with $\|T_\alpha\|, \|T\| \leq r$. By Stone-Weierstrass, for any $\varepsilon > 0$ there is a polynomial $p(z, \bar{z})$ with $\|f(T_\alpha) - p(T_\alpha, T_\alpha^*)\| < \varepsilon$ and $\|f(T) - p(T, T^*)\| < \varepsilon$ on the spectrum in the disk of radius r . Using Proposition 4.2, we have $p(T_\alpha, T_\alpha^*) \rightarrow p(T, T^*)$ in SOT. Therefore, for fixed $x \in H$

$$\|f(T_\alpha)x - f(T)x\| \leq \|f(T_\alpha)x - p(T_\alpha, T_\alpha^*)x\| + \|p(T_\alpha, T_\alpha^*)x - p(T, T^*)x\| + \|p(T, T^*)x - f(T)x\|,$$

and the first and third terms are bounded by $\varepsilon\|x\|$, while the middle term tends to 0. Since ε is arbitrary, $f(T_\alpha)x \rightarrow f(T)x$.

4.2 Cayley transform and self-adjoint case

Define the Cayley transform $u(t) = \frac{t-i}{t+i}$ mapping $\mathbb{R}$ onto the unit circle. For self-adjoint T , $u(T) = (T - iI)(T + iI)^{-1}$ is unitary and depends SOT-continuously on T . This allows one to transfer continuity statements from unitary operators to self-adjoint operators.

Theorem 4.4. *The map $T' \rightarrow (T - iI)(T + iI)^{-1}$ is SOT-continuous from self-adjoint operators to unitary operators.*

Proof. Let $T_\alpha \rightarrow T$ in SOT with all T_α, T self-adjoint. Then $(T_\alpha + iI)^{-1} \rightarrow (T + iI)^{-1}$ in SOT by a resolvent identity and boundedness, and multiplication is jointly SOT-continuous on bounded sets. Hence $(T_\alpha - iI)(T_\alpha + iI)^{-1} \rightarrow (T - iI)(T + iI)^{-1}$ in SOT.

Corollary 4.5. *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and vanishes at infinity, then $T \mapsto f(T)$ is SOT-continuous on the set of self-adjoint operators.*

Proof. Use the Cayley transform to identify $f(T)$ with $g(u(T))$ for a continuous g on the unit circle, and apply SOT-continuity of $u(T)$ and the unitary functional calculus.

4.3 Kaplansky density theorem

Theorem 4.6 (Kaplansky Density Theorem). *Let $A \subseteq B(H)$ be a C^* -subalgebra and let $B = A^{-SOT}$.*

- (a) *The closed unit ball of B is the SOT-closure of the closed unit ball of A .*
- (b) *If $T \in B$ is self-adjoint with $\|T\| \leq 1$, then there exists a net (T_α) of self-adjoint operators in A with $\|T_\alpha\| \leq 1$ and $T_\alpha \rightarrow T$ in SOT.*
- (c) *The self-adjoint part of B is the SOT-closure of the self-adjoint part of A .*

Proof. We prove (b) in detail, then deduce (c) and (a).

Step 1: reduce to self-adjoint approximants. Let $T \in B$ be self-adjoint with $\|T\| \leq 1$. By definition of B there exists a net $S_\alpha \in A$ with $S_\alpha \rightarrow T$ in SOT. Since SOT implies WOT and the adjoint is WOT-continuous on $B(H)$, we have $S_\alpha^* \rightarrow T^* = T$ in WOT. Hence the self-adjoint averages

$$R_\alpha = \frac{1}{2} (S_\alpha + S_\alpha^*) \in A$$

satisfy $R_\alpha \rightarrow T$ in WOT. The set of self-adjoint elements of A is convex, so by Theorem 2.3 the same net can be taken SOT-convergent: there exists a net of self-adjoint elements (still denoted R_α) with $R_\alpha \rightarrow T$ in SOT.

Step 2: cut down the norm using a continuous truncation. Define a continuous real function $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(t) = \begin{cases} -1, & t \leq -1, \\ t, & -1 \leq t \leq 1, \\ 1, & t \geq 1. \end{cases}$$

Then f is bounded by 1 and $f(t) = t$ for $|t| \leq 1$. Applying continuous functional calculus, each $f(R_\alpha)$ is self-adjoint and satisfies $\|f(R_\alpha)\| \leq 1$. Moreover, by Corollary 4.5, $f(R_\alpha) \rightarrow f(T)$ in SOT. Since $\|T\| \leq 1$, we have $f(T) = T$, hence $f(R_\alpha) \rightarrow T$ in SOT with $\|f(R_\alpha)\| \leq 1$. This proves (b) (take $T_\alpha = f(R_\alpha)$).

Step 3: deduce (c). Any self-adjoint $T \in B$ can be scaled to norm ≤ 1 and approximated by (b); rescaling gives a net of self-adjoint elements of A converging to T in SOT.

Step 4: deduce (a). Let $S \in B$ with $\|S\| \leq 1$. Consider the standard 2×2 matrix trick: on $H \oplus H$, the operator

$$X = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$$

X

is self-adjoint with $\|X\| \leq 1$. Applying (b) to the C^* -subalgebra $M_2(A) \subseteq B(H \oplus H)$ and the SOT-closure $M_2(B)$, we obtain a net of self-adjoint matrices in the unit ball of $M_2(A)$ converging to X in SOT. Reading off the $(1, 2)$ -entries yields a net in the unit ball of A converging to S in SOT. Hence the unit ball of A is SOT-dense in the unit ball of B .

Corollary 4.7. *For a C^* -subalgebra $A \subseteq B(H)$, the following are equivalent:*

- (a) *A is a von Neumann algebra.*
- (b) *The closed unit ball of A is WOT-closed.*
- (c) *The closed unit ball of A is SOT-closed.*

Proof. If A is a von Neumann algebra then it is WOT-closed; intersecting with the WOT-compact unit ball of $B(H)$ (Theorem 2.4) gives WOT-closeness of the unit ball. If the unit ball is WOT-closed, then it is SOT-closed because it is convex (Theorem 2.3). Finally, if the unit ball of A is SOT-closed, Kaplansky density (Theorem 4.6) implies A equals its SOT-closure; hence A is a von Neumann algebra. $\square$

5 CONCLUSION

We developed the operator-topological foundations needed to define von Neumann algebras, proved that weak and strong closures coincide on convex sets, established WOT-compactness of the unit ball of $B(H)$, and gave detailed proofs of von Neumann's double commutant theorem and Kaplansky's density theorem. These results form the backbone of the structure theory of von Neumann algebras, leading further to projection comparison, type classification, and modular theory [3, 5].

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